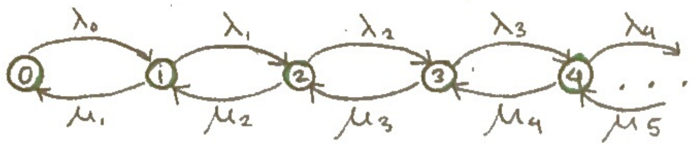
Storgruppsövning 5/12-13

Birth and death processes

$\{X(t); t \geq 0\}$ cont. time, Markov chain with one-step moves.



spends $\exp(\lambda_i + \mu_i)$-dist. time at ⓘ, then moves to $\begin{cases} i+1 \text{ wp } \dfrac{\lambda_i}{\lambda_i+\mu_i} \\ i-1 \text{ wp } \dfrac{\mu_i}{\lambda_i+\mu_i} \end{cases}$

$$P'_0 = G = \begin{pmatrix} -\lambda_0 & \lambda_0 & & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & 0 & & \ddots \end{pmatrix}$$

$$P_t = e^{tG}$$

stat. dist.
$\pi G = 0 \implies \pi_n = \dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \pi_0$

Provided that we can make $\sum_{n=0}^{\infty} \pi_n = 1$ by selecting π_0 appropiately.

Comp. problem

$\lambda_0 = \lambda_1 = \ldots = 1$, $\mu_1 = \mu_2 = \ldots = 2$, $\mu^{(0)} = \pi$ (i.e, $P(X(0)=n) = \pi_n$)

$P\{\max_{0\leq t\leq 10} X(t) \geq 10\} \approx 0{,}00826 \pm 0{,}00003 = \hat{p} \pm \lambda_{0.05}\sqrt{\hat{p}(1-\hat{p})/N}$ where

$\hat{p}$ is the proportion of sample paths that satisfy $\max_{0\leq t\leq 10} X(t) \geq 10$ out of $N = 20\,000\,000$ made.

Solution:

$\pi_n = \left(\frac{1}{2}\right)^n \pi_0$, $n \geq 0$ $= \left(\frac{1}{2}\right)^{n+1}$

P	$X(0)=0$	1	2	3	...	10
	1/2	1/4	1/8	1/16		1/2048

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Rep = 20000000
Sucess = 0

For[ i = 1 : Rep ; Time = 0;
    slump = Rand[0,1]
    if 0 ≤ slump ≤ 1/2   X = 0
    else if 1/2 ≤ slump ≤ 3/4   X = 1
    else if 3/4 ≤ slump ≤ 7/8   X = 2
        ⋮
    else if 1023/1024 ≤ slump ≤ 1   X = 10
    While (X < 10 and Time < 10)
        If X = 0 then X = 1, Time = Time + Rand(exp(1))
          else Time = Time + Rand(exp(3))
               Slump = Rand(0,1)
               if slump ≤ 1/3   x = x+1 else x = x-1.
       if (X = 10 and Time ≤ 10) then Sucess = sucess + 1 ]

Write p = sucess/rep
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6.8.1 and 6.8.2 (G&S)

1. We have a Poisson process $X(t)$ with intensity λ of arriving flies and a Poisson process $Y(t)$ with intensity μ of arriving wasps. Show that $Z(t) = X(t) + Y(t)$ is Poisson process with intensity $\lambda + \mu$.

Solution/proof 1:

$Z(t) - Z(s)$ independent of $(Z(r))_{r \le s}$ $\Longrightarrow$ indep. incr. proc.

$s \le t$ in same way $\Longrightarrow$ stat. incr. proc.

$Z(t)$ is $P_o((\lambda+\mu)t)$ dist. since

$$\Psi_{Z(t)}(\omega) = E(e^{i\omega Z(t)}) = E(e^{i\omega(X(t)+Y(t))}) = E(e^{i\omega(X(t))})E(e^{i\omega(Y(t))}) = e^{\lambda t(e^{i\omega}-1)}e^{\mu t(e^{i\omega}-1)} = e^{(\lambda+\mu)t(e^{i\omega}-1)}$$

∎

Solution/proof 2:

$$P(\min(\exp(\lambda), \exp(\mu)) > z) = P(\exp(\lambda) > z, \exp(\mu) > z) = P(\exp(\lambda) > z)P(\exp(\mu) > z) = e^{-\lambda z}e^{-\mu z} = e^{-(\lambda+\mu)z} = P(\exp(\lambda+\mu) > z)$$

∎

2. Insects land in soup according to Poisson process $X(t)$ with intensity λ. Each such insect is green wp p.
Show that the arrival process $Y(t)$ of green insects into soup is Poisson process with intensity λp.

Solution:

It is enough to check that time $Y(t)$ spends at its values are $\exp(\lambda p)$-dist.

Check PDF of time spent at certain state

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\lambda^n x^{n-1} e^{-\lambda x}(1-p)^{n-1}p = \ldots = \underbrace{\lambda p e^{-\lambda p x}}_{= f_{\exp(\lambda p)}(x)}$$

∎

6.8.5

$B(t)$ is simple birth process with immigration

$\Longleftrightarrow$ birth and death process with $\mu_n = 0$ and $\lambda_n = \lambda n + \nu$

Show that $m(t) = E(B(t))$ satisfies $m'(t) = \lambda m(t) + \nu$ by means of looking at differential-difference equations for $p_n(t) = P(B(t) = n)$.

Also find $m(t)$.

Solution:

$$\frac{d}{dt}\left(e^{-\lambda t}m(t)\right) = m'(t)e^{-\lambda t} - \lambda m(t)e^{-\lambda t} = \nu e^{-\lambda t}$$

$$\Longrightarrow e^{-\lambda t}m(t) = -\frac{\nu e^{-\lambda t}}{\lambda} + C$$

$$m(t) = -\frac{\nu + Ce^{\lambda t}}{\lambda}$$

$$m(0) = 0 \Longrightarrow C = -\nu \Longrightarrow m(t) = \frac{\nu(e^{\lambda t}-1)}{\lambda}$$

$$m(t) = E(B(t)) = \sum_{n=0}^{\infty} nP(B(t)=n) = \sum_{n=0}^{\infty} nP_{0n}(t)$$

Use $P_t' = P_t G$, $\mu^{(t)} = (\mu^{(0)}P_t)' = \mu^{(0)}P_t' = \mu^{(0)}P_t G$ forts.

$$P_{on}(t)' = \left((P_{oo}(t)\ P_{o1}(t), \ldots) \begin{pmatrix} -\nu & \nu & 0 & 0 \\ 0 & -(\lambda+\nu) & \lambda+\nu & 0 \\ 0 & 0 & -(2\lambda+\nu) & (2\lambda+\nu) \\ \vdots & \vdots & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right)_n$$

$n=0: \quad P_{oo}(t)' = -\nu P_{oo}(t)$

$n \geq 1: \quad P_{on}(t)' = (\lambda(n-1)+\nu)P_{on-1}(t) - (\lambda_n+\nu)P_{on}(t)$

$$m'(t) = \sum_{n=0}^{\infty} n P'_{on}(t) = \sum_{n=1}^{\infty} n P_{on}(t) = \sum_{n=1}^{\infty} n\big((\lambda(n-1)+\nu)P_{on-1}(t) - (\lambda n+\nu)p_n(t)\big) =$$

$$= \underbrace{\sum_{n=1}^{\infty} n\nu P_{on-1}(t) - \sum_{n=1}^{\infty} n\nu P_{on}(t)}_{\underbrace{\sum_{n=1}^{\infty}\nu P_{on-1}(t) + \sum_{n=1}^{\infty}(n-1)\nu P_{on-1}(t) - \sum_{n=0}^{\infty} n\nu P_{on}(t)}_{\nu}} + \underbrace{\underbrace{\sum_{n=2}^{\infty} n(n-1)\lambda P_{on-1}(t)}_{} - \underbrace{\sum_{n=1}^{\infty} n^2\lambda P_{on}(t)}_{}}_{\underbrace{\sum_{n=1}^{\infty}(n+1)n\lambda P_{on}(t) - \sum_{n=1}^{\infty} n^2\lambda P_{on}(t)}_{= \sum_{n=1}^{\infty}\lambda n P_{on}(t) = \lambda m(t)}}$$

$= \nu + \lambda m(t).$

6.8.6

Let $N(t)$ be birth process with $N(0)=0$. Find $P_{on}(t) = P(N(t)=n)$.

Solution:

During lecture time previous week we established that

$\hat{P}_{on}(\theta) = \int_0^{\infty} e^{-\theta t} P_{on}(t)\,dt$ satisfies $\hat{P}_{on}(\theta) = \frac{1}{\lambda_n}\frac{\lambda_0}{\theta+\lambda_0}\cdots\frac{\lambda_n}{\theta+\lambda_n}$

where $\frac{\lambda}{\theta+\lambda} = \int_0^{\infty} e^{-\theta\lambda}\,\lambda e^{-\lambda t}\,dt$ $\qquad = \sum_{k=0}^{n} \frac{a_k\lambda_k}{\theta+\lambda_k}$

$$\Rightarrow P_{on}(t) = \sum_{k=0}^{n} a_k\lambda_k e^{-\lambda_k t}$$

6.9.9

Let i be a transient state of a continous time Markov chain $X(t)$ with $X(0)=i$. Show that the total time spent at i has an exponential distribution

Solution:

Chain transient, means that there is certain probability $p>0$ to escape from i at each start in i. Probability is $(1-p)^{n-1}p$ that we escape from i forever at attempt no n.

Now the PDF of the time it takes to escape from i is calculated as in 6.8.2, because each time spent away from i given that we come back will be exp.-dist.

6.9.10

Let $X(t)$ be an assymetric simple random walk in continous time on the non-negative integers, meaning birth-death-process with $\lambda_n = \lambda$ and $\mu_n = \mu$. Assuming that $\lambda > \mu$, show that total time spent at state r is exponentially distributed with parameter $\lambda - \mu$.

Solution:

Each state r is transient and therefor it is same calculation as prevous exercise. Special case of 6.9.9.

6.9.1

Continous time Markov chain with values $\{1,2\}$ and $G = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$

Calculate $P_t = e^{tG} = \sum_{n=0}^{\infty} \frac{(tG)^n}{n!}$.

Solve $\pi G = 0$ to find stat. dist. and check that $\lim_{t\to\infty} P_{ij}(t) = \pi_j$ (as is claimed by convergence theorem).

Solution:

$$\begin{cases} \pi G = 0 \\ \pi_0 + \pi_1 = 1 \end{cases} \Longrightarrow \pi = \left(\frac{\lambda}{\lambda+\mu} \quad \frac{\mu}{\lambda+\mu} \right)$$

$$G = B \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B^{-1} \Longrightarrow G^n = B \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} B^{-1}$$

$$e^{tG} = B \begin{pmatrix} e^{\lambda_1 t} & 1 \\ 1 & e^{\lambda_2 t} \end{pmatrix} B^{-1}$$

$x = \lambda_1, \lambda_2$

λ_1 and λ_2 must be eigenvalues of G, i.e $\det(G - xI) = 0$. and $B = (b_1\ b_2)$ where b_i are columnmatrices that satisfy $Gb_i = \lambda_i b_i$ i.e b_1 and b_2 are the corresponding eigenvectors.

$$\det(G - xI) = \begin{vmatrix} -(\mu+x) & \mu \\ \lambda & -(\lambda+x) \end{vmatrix} = (\lambda+x)(\mu+x) - \lambda\mu = x(x+\mu+\lambda) = 0$$

$$\Longrightarrow \lambda_1 = 0, \quad \lambda_2 = -(\lambda+\mu).$$

$$B = \begin{pmatrix} 1 & \mu \\ 1 & -\lambda \end{pmatrix} \qquad G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad G\begin{pmatrix} \mu \\ -\lambda \end{pmatrix} = \begin{pmatrix} -(\mu^2 + \mu\lambda) \\ \mu\lambda + \lambda^2 \end{pmatrix} = -(\lambda+\mu)\begin{pmatrix} \mu \\ \lambda \end{pmatrix}$$

$$B^{-1} = \frac{1}{\lambda+\mu} \begin{pmatrix} \lambda & \mu \\ 1 & -1 \end{pmatrix}$$

$$e^{tG} = \begin{pmatrix} 1 & \mu \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & e^{-(\lambda+\mu)t} \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ 1 & -1 \end{pmatrix} \frac{1}{\lambda+\mu} = \frac{1}{\lambda+\mu} \begin{pmatrix} \lambda + \mu e^{-(\lambda+\mu)t} & \mu - \mu e^{-(\lambda+\mu)t} \\ \lambda - \lambda e^{-(\mu+\lambda)t} & \mu + \lambda e^{-(\lambda+\mu)t} \end{pmatrix}$$

$$\xrightarrow[\text{as } t\to\infty]{} \frac{1}{\lambda+\mu} \begin{pmatrix} \lambda & \mu \\ \lambda & \mu \end{pmatrix}$$

6.9.2 a)

$P(X(t)=2 \mid X(0)=1, X(3t)=1)$ for chain in 6.9.1

$$P(X(t)=2 \mid X(0)=1, X(3t)=1) = \frac{P(X(0)=1, X(t)=2, X(3t)=1)}{P(X(0)=1, X(3t)=1)} =$$

$$= \frac{(\cancel{\mu^{(0)}})_1 P_{12}(t) P_{21}(2t)}{(\cancel{\mu^{(0)}})_1 P_{11}(3t)} = \left\{ \begin{array}{c} \text{insert answer} \\ \text{from 6.9.1} \end{array} \right\} = \text{answer.}$$